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கல்வி அமைச்சு
Ministry of Education



# G.C.E.(A.L) Support Seminar - 2023

සංයුක්ත ගණිතය I
இணைந்த கணிதம் I
Combined Mathematics I

10 E I

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மூன்று மணித்தியாலம்
Three hours

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மேலதிக வாசிப்பு நேரம் - 10 நிமிடங்கள்
Additional Reading Time - 10 minutes

**Use additional reading time to go through the question paper, select the questions you will answer and decide which of them you will prioritise.**

| Index Number | | | | | | | |
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## Instructions:

- *This question paper consists of two parts;*
  **Part A** (Questions 1 – 10) *and* **Part B** (Questions 11 – 17)
- **Part A:**
  *Answer **all** questions. Write your answers to each question in the space provided. You may use additional sheets if more space is needed.*
- **Part B:**
  *Answer **five** questions only. Write your answers on the sheets provided.*
- *At the end of the time allotted, tie the answer scripts of the two parts together so that* **Part A** *is on top of* **Part B** *and hand them over to the supervisor.*
- *You are permitted to remove **only** **Part B** of the question paper from the Examination Hall.*

**For Examiners' Use only**

| (10) Combined Mathematics I | | |
|---|---|---|
| **Part** | **Question No.** | **Marks** |
| A | 1 | |
| | 2 | |
| | 3 | |
| | 4 | |
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| Supervised by: | | |

[see page two

**Part A**

**1.** Using the **principal of Mathematical induction** prove that $\sum_{r=1}^{n} 2^r = 2(2^n - 1)$ for all $n \in \mathbb{Z}^+$

**2** Sketch the graph of $= \left||x-2|-2\right|$ . Hence or otherwise solve the equation $\left||x-2|-2\right| = \frac{x}{2}$

[see page three

**3** Shade the region $R$ that represents the complex number $Z$ satisfying the condition $0 \leq ArgZ \leq \frac{\pi}{3}$ in an Argand Diagram.
Also find the least Value of $|iZ + 2|$ in the region $R$.

**4** **Write down** the binomial expansion of $(1 + x)^n$ in ascending powers of $x$. Given that the coefficient of $x^2$ in the expansion $(1 + x + ax^2)^7$ is 14. Show that $a = -1$.

[see page four

**5** Show that $\lim_{x\to\frac{\pi}{4}} \frac{2\sqrt{x}-\sqrt{\pi}}{\sin(x-\frac{\pi}{4})} = \frac{2}{\sqrt{\pi}}$

**6** If $f(x) = (x+1)\tan^{-1}\sqrt{x} - \sqrt{x}$ , then find $\frac{d[f(x)]}{dx}$. Hence, deduce $\int \tan^{-1}\sqrt{x}\,dx$. The region enclosed by the curves $y = \sqrt{\tan^{-1}\sqrt{x}}$ , $x = 3$ and $y = 0$ is rotated about the $x-$ axis through $2\pi$ radians. Show that the volume of the solid thus generated is $\frac{\pi}{3}\left(4\pi - 3\sqrt{3}\right)$ cubic units.

[see page five

**7** A curve C is given by the parametric equations $x = a\cos\theta$ and $y = b\sin\theta$ for $(0 \leq \theta \leq \pi)$. Show that the equation of the normal to the curve C, at point P, is $ax\sec\alpha\ - \text{b}y\text{cosec}\,\alpha\ + b^2 - a^2 = 0$ .

Also find the normal to the curve C, at point $\left(-\frac{a}{\sqrt{2}}, \frac{b}{\sqrt{2}}\right)$ on the curve C.

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**8** The straight line $l \equiv y - mx = 0$ passes through the point of intersection of two straight lines $4x + 3y - k = 0$, where $k$ is constant and $5x - 12y + 7 = 0$. Find the value of $m$ in terms of $k$. Further, given that the line, $l = 0$ is perpendicular to the line $x + y = 0$. Find the values of $m$ and $k$.

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[see page six

**9** A circle $S$ with centre on the y-axis intersects the circle $x^2 + y^2 = 9$ orthogonally and the circle $x^2 + y^2 + x - 7y + 5 = 0$ bisects the circumference of the circle $S$. Show that there are two such circle $S$, and find their equations.

**10** Given that $tanA = \frac{5}{12}$ and $sinB = \frac{4}{5}$; Where A and B are such that $\pi < A < \frac{3\pi}{2}$ and $\frac{\pi}{2} < B < \pi$. Find the value of $sin(A + B)$

[see page seven

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10 S I

**Part B**

✱ Answer **five** questions only.

**11 (a)** Write down the sum and the product of the roots of quadratic equation $ax^2 + bx + c = 0$, in terms of $a, b$ and $c$ where $a, b, c \in \mathbb{R}, a \neq 0$

Given that $f(x) \equiv x^2 - p^2qx + q^2$ where $p, q \in \mathbb{R}^+$ and roots of the equation $f(x) = 0$ are α and β.

(i) Find $\alpha^{\frac{3}{2}} + \beta^{\frac{3}{2}}$ in terms of $p$ and $q$.

(ii) If α and β are real, then find the least integer value of $p$.

(iii) For the above $p$ value find the quadratic equation in terms of $q$, whose roots are $\alpha^{\frac{3}{2}}$ and $\beta^{\frac{3}{2}}$.

**(b)** Let $P(x) \equiv 2x^3 + x^2 - 2x + \lambda$ ; where $\lambda \in \mathbb{R}^+$

(i) If $\lambda$ is zero of the polynomial $P(x)$, find $\lambda$

(ii) If $-\lambda$ is zero of the polynomial $P(x)$, find $\lambda$

(iii) For the value of $\lambda$ which satisfies both (i) and (ii), write down the polynomial $P(x)$ and express $P(x)$ as a multiple of linear factors.

(iv) Find the remainder, when $P(x) + 3x + 2$ is divided by $x^2 + 1$.

**12 (a)** An institution has 8 cars and there are parking facilities in two rows, 4 cars in each row in the park.

(i) Find the number of ways in which 8 cars can be parked.

(ii) Find the number of ways in which 8 cars can be parked, if the first place in the first row is to be reserved for chairman's car and a place in the first row for the car of secretaries.

(iii) If the first place in the first row should be given to either one of the two cars of chairman's or secretaries, and if the other car should also have a place in the first row, find the number of ways in which the cars can be parked.

[see page eight

**(b)** Find the value of the constants A and B such that,

$$\frac{r^2+3r-1}{(r^2-r+1)(r^2+r+1)} = \frac{Ar+B}{r^2-r+1} - \frac{Ar+2B}{r^2+r+1} \quad ; \text{where } r\epsilon\mathbb{Z}^+.$$

If, $U_r = \frac{r^2+3r-1}{(r^2-r+1)(r^2+r+1)}$ then determine $f_r$ such that $U_r = f_r - f_{r+1}$

Hence, show that $\sum_{r=1}^{n} U_r = 2 - \frac{(n+2)}{n^2+n+1}$

Is this series convergent? Justify your answer.

If, $\sum_{r=1}^{n} U_r < 2 - \frac{11}{91}$ then find greatest value of $n$.

**13 (a)** If $A = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}$, show that any real matrix B which commutates with A, under multiplication, can be written in the form $\lambda A + \mu I$, where $\lambda$ *and* $\mu$ are real numbers and $I$ is the identity matrix of order 2. Find the value of $\lambda$ *and* $\mu$ *when* $B = A^2$ Hence Find $A^{-1}$.

**(b)** By Factorizing $Z^6$-1, completely solve the equation $Z^6 = 1$.

If $Z_1$ and $Z_2$ are any two distinct roots of the equation $Z^6 = 1$, show by reference to an Argent diagram, or otherwise, that the three possible values of $|Z_1 - Z_2|$ are 1, 2 and $\sqrt{3}$.

**(c)** By using De moivre's theorem for positive integer $n$,

Show that $\left(\frac{1+\sin\theta+i\cos\theta}{1+\sin\theta-i\cos\theta}\right)^n = \cos n\left(\frac{\pi}{2}-\theta\right) + i\sin n\left(\frac{\pi}{2}-\theta\right)$

Deduce that, $\left(\frac{1+i}{1-i}\right)^{2n} = (-1)^n$

**14 (a)** Let $f(x) = \frac{x(x+3)}{(x+1)^2}$ for $x \neq -1$

Show that $f'(x)$ the first derivative of $f(x)$ with relative to $x$, is given by

$f'(x) = -\frac{(x-3)}{(x+1)^3}$

.

Hence, find the intervals on which $f(x)$ is decreasing and the intervals on which $f(x)$ is increasing.

Obtain the coordinates of the turning point of $f(x)$.

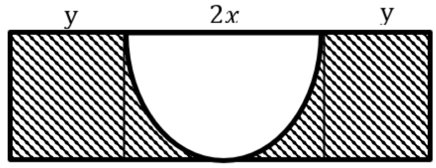
It is given that $f''(x) = \frac{2(x-5)}{(x+1)^4}$ for $x \neq -1$.

Find the coordinates of the point of inflection on the graph of $y = f(x)$.

Sketch the graph of $y = f(x)$ indicating the asymptotes, turning point and point of inflection.

[see page nine

**(b)** The shaded region shown in the figure is obtained by removing a semicircular lamina of radius $x\ m$ from a rectangle of length $2(x+y)\ m$ and width $x\ m$.




The area of the rectangle is $8\pi\ m^2$. Show that the perimeter p of the shaded region, measured in meters, is given by $P = \pi\left(x + \frac{16}{x}\right)$

**15** **(a)** Determine the values of constants A, B and C such that

$x^4 + 1 = A(x^4 - 1) + B(x^2 + 1)(x + 1) + C(x^2 + 1)(x - 1) - (x^2 - 1)$ for $x \in \mathbb{R}$ ,

hence, find $\int \frac{x^4+1}{x^4-1} dx$

**(b)** **(i)** If $y = x + \cos x \sin^3 x$ show that $\frac{dy}{dx} = 1 + 3\sin^2 x - 4\sin^4 x.$

Given that $I = \int_0^{\frac{\pi}{2}} (x + 3x\sin^2 x - 4x\sin^4 x)\, dx$. By using above result and using integration by parts, Show that $I = \frac{1}{8}(\pi^2 - 2)$

**(ii)** Further given that,

$J_1 = \int_0^{\frac{\pi}{2}} (1 + 3\cos^2 x - 4\cos^4 x)\, dx$

$J_2 = \int_0^{\frac{\pi}{2}} (x + 3x\cos^2 x - 4x\cos^4 x)\, dx$

Using the result $\int_0^a f(x)\, dx = \int_0^a f(a-x)\, dx$

Show that $I = \frac{\pi}{2}J_1 - J_2$

Now given that $\frac{d}{dx}(x - \sin x \cos^3 x) = 1 + 2\cos^2 x - 4\cos^4 x$

show that $J_2 = \frac{1}{8}(\pi^2 + 2)$, deduce the value of $J_1$.

**(c)** Using the substitution $\sqrt{x^3 + 1} = t$ , Evaluate $\int_0^2 \frac{x^8}{\sqrt{x^3+1}} dx.$

[see page ten

**16** $l_1: x - \sqrt{3}y + 1 + k = 0$ and $l_2: x + \sqrt{3}y + 1 - k = 0$ are two given straight lines passing through the point $(-1, 3)$ show that $k = 3\sqrt{3}$.

For that value of k, find the equations of the angle bisectors between the straight lines $l_1 = 0$ and $l_2 = 0$.

Let, $l$ be the acute angle bisector of $l_1$ and $l_2$. Show that the point $A \equiv (2,3)$ lies on the line $l = 0$.

Find the equation of the circle $S$ with centre $A$ and the length of the diameter is 3 units.

Find the perpendicular distance from the point $A$ to the line $l_1 = 0$, hence find the equation of the tangent drawn from $(-1, 3)$ to the circle $S$.

From a point $P$ on the line $l = 0$, two tangents are drawn to the circle $S$ so that they are perpendicular to each other.

Show that there are two such points for $P$ and in each case find the coordinates.

Further, find the area of the quadrilateral which enclosed by the tangents.

**17 (a) (i)** Write down $\cos(A + B)$ In terms of $\cos A, \cos B, \sin A, \sin B$ and obtain an expression for $\cos 3A$ in terms of $\cos A$.

**(ii)** Determine constants $\lambda$ and $k$ such that,

$$\frac{2\cos 3x - 4cos^5x + 3cos^3x}{cosx(1 + sin^2x)} = \lambda \cos 2x + k$$

Hence, find the maximum and minimum values of

$$f(x) = \frac{2\cos 3x - 4cos^5x + 3cos^3x}{cosx(1 + sin^2x)}$$

and sketch the graph of $y = f(x)$ for $x\epsilon[-\pi, \pi]$

**(b)** A point $P$ is inside the triangle ABC, such that $P\hat{A}B = P\hat{B}C = P\hat{C}A = \alpha$

By applying **Sine Rule** for suitable triangles, write down two expressions for $PC_1$ and show that $cot\alpha = cotA + cotB + cotC$

**(c)** Solve the equation $2tan^{-1}(cosx) = tan^{-1}(2cosecx)$ for $x\epsilon\left(0, \frac{\pi}{2}\right)$.

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